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An Inspection on the Homogeneous Sexnary Cubic Equation

$$(w^2+p^2-z^2)(w-p)=(1+2k^2)(x+y)R^2$$

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ABSTRACT

This paper aims at finding non-zero distinct integer solutions to homogeneous cubic Diophantine equation with six unknowns given by $(w^2+p^2-z^2)(w-p)=(1+2k^2)(x+y)R^2$. The substitution technique and factorization method are utilized to obtain the required integer solutions to the equation of degree three with six unknowns given in title.

KEYWORDS: Homogeneous cubic equation , Sexanary cubic equation ,Integer solutions, Factorization method ,Substitution technique

Mathematical Subject Classification:11 D 25

1. INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of cubic

Diophantine equations with multi variables in [1-17]. This paper aims at determining many integer solutions to homogeneous polynomial equation of degree three with six unknowns given by $(w^2+p^2-z^2)(w-p)=(1+2k^2)\,(x+y)R^2$.

2. METHOD OF ANALYSIS

The homogeneous cubic equation with six unknowns to be solved is

$$(w^2+p^2-z^2)(w-p)=(1+2k^2)\,(x+y)R^2 \quad (2.1)$$

Introduction of the linear transformations

$$x=v+1, y=v-1, z=u, w=u+v, p=u-v, u\neq v\neq \pm 1 \quad (2.2)$$

in (2.1) leads to

$$u^2+2v^2=(1+2k^2)R^2 \quad (2.3)$$

The above equation (2.3) is solved through different ways and thus ,one obtains

different sets of integer solutions to (2.1).

Set 2. 1:

It is seen that (2.3) is satisfied by

$$u=(1+2k^2), v=k(1+2k^2), R=(1+2k^2) \quad (2.4)$$

In view of (2.2),the corresponding integer solutions to (2.1) are given by

$$x=k(1+2k^2)+1, y=k(1+2k^2)-1, z=(1+2k^2),$$
$$w=(k+1)(1+2k^2), p=(1-k)(1+2k^2)$$

jointly with R given in (2.4) .

Set 2. 2:

Write (2.3) as

$$(1+2k^2)R^2-2v^2=u^2=u^2*1 \quad (2.5)$$

Assume

$$u=(1+2k^2)a^2-2b^2 \tag{2.6}$$

Write the integer 1 on the R.H.S. of (2.5) as

$$1=(\sqrt{1+2k^2}+\sqrt{2}k)(\sqrt{1+2k^2}-\sqrt{2}k) \tag{2.7}$$

Substituting (2.6) & (2.7) in (2.5) and employing the method of factorization , consider

$$\sqrt{1+2k^2}R+\sqrt{2}v=(\sqrt{1+2k^2}+\sqrt{2}k)(\sqrt{1+2k^2}a+\sqrt{2}b)^2$$

Equating the coefficients of corresponding terms ,note that

$$R=(1+2k^2)a^2+2b^2+4abk,\ v=k[(1+2k^2)a^2+2b^2]+2(1+2k^2)ab \tag{2.8}$$

Using (2.2),the corresponding integer solutions to (2.1) are as below:

$$\begin{aligned}
x&=k[(1+2k^2)a^2+2b^2]+2(1+2k^2)ab+1,\\
y&=k[(1+2k^2)a^2+2b^2]+2(1+2k^2)ab-1,\\
z&=(1+2k^2)a^2-2b^2,\\
w&=(1+k)(1+2k^2)a^2+2b^2(k-1)+2(1+2k^2)ab,\\
p&=(1-k)(1+2k^2)a^2-2b^2(k+1)-2(1+2k^2)ab,\\
R&=(1+2k^2)a^2+2b^2+4abk
\end{aligned}$$

Set 2. 3:

Write (2.3) as

$$(1+2k^2)R^2-u^2=2v^2 \tag{2.9}$$

Assume v as

$$v=(1+2k^2)a^2-b^2 \tag{2.10}$$

Write the integer 2 on the R.H.S. of (2.9) as

$$2=\frac{(\sqrt{1+2k^2}+1)(\sqrt{1+2k^2}-1)}{k^2} \tag{2.11}$$

Following the procedure as in Way 2,the corresponding integer solutions to (2.1)

are given by

$$x = k^2[(1+2k^2)A^2 - B^2]+1, y = k^2[(1+2k^2)A^2 - B^2]-1,$$
$$z = k[(1+2k^2)A^2 + B^2 + 2(1+2k^2)AB],$$
$$w = k(k+1)(1+2k^2)A^2 + k(1-k)B^2 + 2(1+2k^2)kAB,$$
$$p = k(1-k)(1+2k^2)A^2 + k(k+1)B^2 + 2(1+2k^2)kAB,$$
$$R = k[(1+2k^2)A^2 + B^2 + 2AB].$$

Set 2. 4:

Taking

$$v = kV, k > 1 \tag{2.12}$$

in (2.3) ,it is written as

$$u^2 = (1+2k^2)R^2 - 2k^2V^2 \tag{2.13}$$

Introducing the linear transformations

$$R = X + 2k^2T, V = X + (1+2k^2)T \tag{2.14}$$

in (2.13), it is written as

$$X^2 = 2k^2(1+2k^2)T^2 + u^2 \tag{2.15}$$

which is equivalent to the system of double equations as shown below:.

Case (a):

$$X + u = (1+2k^2)T$$
$$X - u = 2k^2T$$

Solving these two linear equations, we get

$$X = \frac{(1+4k^2)T}{2},$$
$$u = T$$

Taking $T = 2s$, we get the integer solution of X and u as

$$X=(1+4k^2)s,$$
$$u=2s \qquad (2.16)$$

Substituting the values of X , T in (14) , we get

$$R=(1+8k^2)\,s, V=(3+8k^2)\,s \qquad (2.17)$$

and from (2.12) , it is seen that

$$v=k(3+8k^2)\,s \qquad (2.18)$$

Using the values of u ,v from (2.16) & (2.18) in (2.2) ,the integer solutions to (2.1) are giv4en by

$$x=k(3+8k^2)s+1,$$
$$y=k(3+8k^2)s-1,$$
$$z=2s,$$
$$R=(8k^2+1)s,$$
$$w=(8k^3+3k+2)s,$$
$$p=(2-8k^3-3k)s.$$

Case (b):

$$X+u=(4k^2+2)\,T$$
$$X-u=k^2\,T$$

Solving these two linear equations, we get

$$X=\frac{(5k^2+2)\,T}{2},$$
$$u=\frac{(3k^2+2)T}{2}$$

Taking $T=2s$, we get the integer solution of X and u as

$$X=(5k^2+2)s,$$
$$u=(3k^2+2)s \qquad (2.19)$$

Substituting the values of X, T in (2.14) , we get

$$R=(2+9k^2)\,s, V=(4+9k^2)\,s \qquad (2.20)$$

and from (2.12) ,it is seen that

$$v = k(4+9k^2)\, s \tag{2.21}$$

Using the values of u ,v from (2.19) & (2.21) in (2.2) ,the integer solutions to (2.1) are giv4en by

$$\begin{aligned} x &= k(4+9k^2)s+1, \\ y &= k(4+9k^2)s-1, \\ z &= (2+3k^2)s, \\ R &= (9k^2+2)s\ , \\ w &= (9k^3+3k^2+4k+2)s, \\ p &= (2-9k^3+3k^2-4k)s. \end{aligned}$$

Case (c):

$$\begin{aligned} X + u &= (4k^3+2k)\ T \\ X - u &= kT \end{aligned}$$

Solving these two linear equations, we get

$$X = \frac{(4k^3+3k)\ T}{2}\ ,$$

$$u = \frac{(4k^3+k)T}{2}$$

Taking $T = 2s$, we get the integer solution of X and u as

$$\begin{aligned} X &= (4k^3+3k)s, \\ u &= (4k^3+k)s \end{aligned} \tag{2.22}$$

Substituting the values of X, T in (2.14) , we get

$$R = (4k^3+4k^2+3k)\ s, V = (4k^3+4k^2+3k+2)\ s \tag{2.23}$$

and from (2.12) ,it is seen that

$$v = k(4k^3+4k^2+3k+2)\ s \tag{2.24}$$

Using the values of u ,v from (2.22) & (2.24) in (2.2) ,the integer solutions to (2.1) are giv4en by

$$x = k(4k^3 + 4k^2 + 3k + 2)s + 1,$$
$$y = k(4k^3 + 4k^2 + 3k + 2)s - 1,$$
$$z = (k + 4k^3)s,$$
$$R = (4k^3 + 4k^2 + 3k)s,$$
$$w = (4k^4 + 8k^3 + 3k^2 + 3k)s,$$
$$p = (-4k^4 - 3k^2 - k)s.$$

Case (d):

$$X + u = (4k^4 + 2k^2)\, T$$
$$X - u = T$$

Solving these two linear equations, we get

$$X = \frac{(4k^4 + 2k^2 + 1)\, T}{2},$$
$$u = \frac{(4k^4 + 2k^2 - 1)T}{2}$$

Taking $T = 2s$, we get the integer solution of X and u as

$$X = (4k^4 + 2k^2 + 1)s,$$
$$u = (4k^4 + 2k^2 - 1)s \qquad (2.25)$$

Substituting the values of X, T in (2.14) , we get

$$R = (4k^4 + 6k^2 + 1)\, s, V = (4k^4 + 6k^2 + 3)\, s \qquad (2.26)$$

and from (2.12) ,it is seen that

$$v = k(4k^4 + 6k^2 + 3)\, s \qquad (2.27)$$

Using the values of u ,v from (2.25) & (2.27) in (2.22) ,the integer solutions to (2.1) are giv4en by

$$x = k(4k^4 + 6k^2 + 3)s + 1,$$
$$y = k(4k^4 + 6k^2 + 3)s - 1,$$
$$z = (2k^2 + 4k^4 - 1)s,$$
$$R = (4k^4 + 6k^2 + 1)s,$$
$$w = (4k^5 + 4k^4 + 6k^3 + 2k^2 + 3k - 1)s,$$
$$p = (-4k^5 + 4k^4 - 6k^3 + 2k^2 - 3k - 1)s.$$

Set 2. 5:

Let

$$R = a^2 + 2b^2 \tag{2.28}$$

Write $(1 + 2k^2)$ as

$$(1 + 2k^2) = (1 + i\sqrt{2}k)(1 - i\sqrt{2}k) \tag{2.29}$$

Using (2.28) and (2.29) in (2.3) and employing the method of factorization, define

$$\begin{aligned} u + i\sqrt{2}v &= (1 + i\sqrt{2}k)\,(a + i\sqrt{2}b)^2 \\ &= (1 + i\sqrt{2}k)\,[f(a,b) + i\,\sqrt{2}\,g(a,b)] \end{aligned} \tag{2.30}$$

where

$$f(a,b) = (a^2 - 2b^2)\,, g(a,b) = 2ab$$

Equating the real and imaginary parts of (2.30),we get

$$u = f(a,b) - 2k\,g(a,b) = (a^2 - 2b^2) - 4abk$$
$$v = k\,f(a,b) + g(a,b) = k(a^2 - 2b^2) + 2ab$$

Substituting the values of u and v in (2.2), the non-zero distinct integral solutions of (2.1) are given by

$$x = k(a^2 - 2b^2) + 2ab + 1$$
$$y = k(a^2 - 2b^2) + 2ab - 1$$
$$z = a^2 - 2b^2 - 4abk$$
$$w = (a^2 - 2b^2)(k + 1) + 2ab\,(1 - 2k)$$
$$p = (a^2 - 2b^2)(1 - k) - 2ab\,(1 + 2k)$$

jointly with R given in (2.28) .

Set 2. 6

Consider (3) as

$$u^2 + 2v^2 = (1+2k^2)\ R^2 * 1 \qquad (2.31)$$

Write the integer 1 in (2.31) as

$$1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9} \qquad (2.32)$$

Assume

$$R = 9\ (a^2 + 2b^2) \qquad (2.33)$$

Substituting (2.29) ,(2.32) & (2.33) in (2.31) and applying factorization , consider

$$u + i\sqrt{2}\,v = (1+i\sqrt{2}\,k)\ 9(a+i\sqrt{2}\,b)^2 \frac{(1+i2\sqrt{2})}{3}$$
$$= 3[1-4k+i\sqrt{2}(2+k)]\,[f(a,b)+i\sqrt{2}\,g(a,b)]$$

Equating the coefficients of corresponding terms , we have

$$u = 3[(1-4k)f(a,b) - 2(2+k)g(a,b)],$$
$$v = 3[(2+k)f(a,b) + (1-4k)g(a,b)].$$

In view of (2.2) , the integer solutions to (2.1) are as shown below:

$$x = 3[(2+k)f(a,b) + (1-4k)g(a,b)] + 1,$$
$$y = 3[(2+k)f(a,b) + (1-4k)g(a,b)] - 1,$$
$$z = 3[(1-4k)f(a,b) - 2(2+k)g(a,b)],\quad j$$
$$w = 3[(3-3k)f(a,b) - (3+6k)\ g(a,b)],$$
$$p = 3[(-1-5k)f(a,b) + (2k-5)g(a,b)]$$

jointly with R given by (2.33) .

Note 1

Apart from (32) , the integer 1 may be expressed as below:

$$1=\frac{(2r^2-s^2+i\sqrt{2}(2rs))\,(2r^2-s^2-i\sqrt{2}(2rs))}{(2r^2+s^2)^2},$$

$$1=\frac{(r^2-2s^2+i\sqrt{2}(2rs))\,(r^2-2s^2-i\sqrt{2}(2rs))}{(r^2+2s^2)^2},$$

$$1=\frac{(2+ig_n)\,(2-ig_n)}{(f_n)^2},n=0,1,2,\ldots$$

where

$$f_n=(3+2\sqrt{2})^{n+1}+(3-2\sqrt{2})^{n+1},$$

$$g_n=(3+2\sqrt{2})^{n+1}-(3-2\sqrt{2})^{n+1}.$$

Following the above procedure , three more sets of integer solutions to (2.1) are obtained.

Set 2. 7

Rewrite (3) as

$$(u+R)\,(u-R)=2\,(kR+v)(kR-v)$$

The above equation can be written in the form of ratio as

$$\frac{(u+R)}{(kR+v)}=\frac{2\,(kR-v)}{(u-R)}=\frac{\alpha}{\beta},\beta\neq 0 \tag{2.34}$$

which is equivalent to the system of double equations

$$u\,\beta-v\,\alpha+R(\beta-k\alpha)=0$$

$$u\,\alpha+2\,v\,\beta-R\,(2k\,\beta+\alpha)=0$$

Employing the method of cross- multiplication, we get

$$\begin{aligned}u&=\alpha^2-2\,\beta^2+4\,\alpha\,\beta k\\ v&=-k\alpha^2+2k\,\beta^2+2\,\alpha\,\beta\\ R&=\alpha^2+2\,\beta^2\end{aligned} \tag{2.35}$$

Using the above values in (2.2), we get the corresponding non-zero integer solutions of (2.1) to be

$$x = -k\alpha^2 + 2k\beta^2 + 2\alpha\beta + 1$$
$$y = -k\alpha^2 + 2k\beta^2 + 2\alpha\beta - 1$$
$$z = \alpha^2 - 2\beta^2 + 4k\alpha\beta$$
$$w = (1-k)\alpha^2 + 2(k-1)\beta^2 + (4k+2)\alpha\beta$$
$$p = (k+1)\alpha^2 - (2k+2)\beta^2 + (4k-2)\alpha\beta$$

jointly with the value of R in (2.35) .

Note 2

Equation (2.3) may also be expressed in the form of ratios as follows:

$$\text{(i)} \quad \frac{(u+R)}{2(kR+v)} = \frac{(kR-v)}{(u-R)} = \frac{\alpha}{\beta}$$
$$\text{(ii)} \quad \frac{(u+R)}{(kR-v)} = \frac{2(kR+v)}{(u-R)} = \frac{\alpha}{\beta}$$

Repeating the above process as in Way 7, we get the corresponding integer solutions to (2.1) .

Set 2. 8

Consider (2.3) as

$$2v^2 + u^2 = (2k^2+1)R^2 \qquad (*)$$

Let

$$R = 2a^2 + b^2 \qquad (2.36)$$

Write $(2k^2+1)$ as

$$(2k^2+1) = (\sqrt{2}k + i)(\sqrt{2}k - i) \qquad (2.37)$$

Using (2.36) and (2.37) in (*) and employing the method of factorization, define

$$\begin{aligned}\sqrt{2}v + iu &= (\sqrt{2}k+i)(\sqrt{2}a+ib)^2 \\ &= (\sqrt{2}k+i)[F(a,b) + i\sqrt{2}g(a,b)]\end{aligned} \qquad (2.38)$$

where

$$F(a,b) = (2a^2 - b^2)$$

Equating the real and imaginary parts of (2.38),we get

$$u = F(a,b) + 2k\,g(a,b) = (2a^2 - b^2) + 4abk$$
$$v = k\,F(a,b) - g(a,b) = k(2a^2 - 2b^2) - 2ab$$

Substituting the values of u and v in (2.2), the non-zero distinct integral solutions of (2.1) are given by

$$x = k(2a^2 - b^2) - 2ab + 1$$
$$y = k(2a^2 - b^2) - 2ab - 1$$
$$z = 2a^2 - b^2 + 4abk$$
$$w = (2a^2 - b^2)(k+1) + 2ab\,(2k-1)$$
$$p = (2a^2 - b^2)(1-k) + 2ab\,(1+2k)$$

jointly with R given in (2.36) .

3. CONCLUSION

The technical procedure to obtain integer solutions for the homogeneous sexanary cubic equation given by $(w^2 + p^2 - z^2)(w-p) = (1+2k^2)(x+y)R^2$ is illustrated in this paper. As the cubic equations are plenty ,one may search integer solutions to other choices of cubic equations with multiple variables.

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